3. Points in the coordinate system

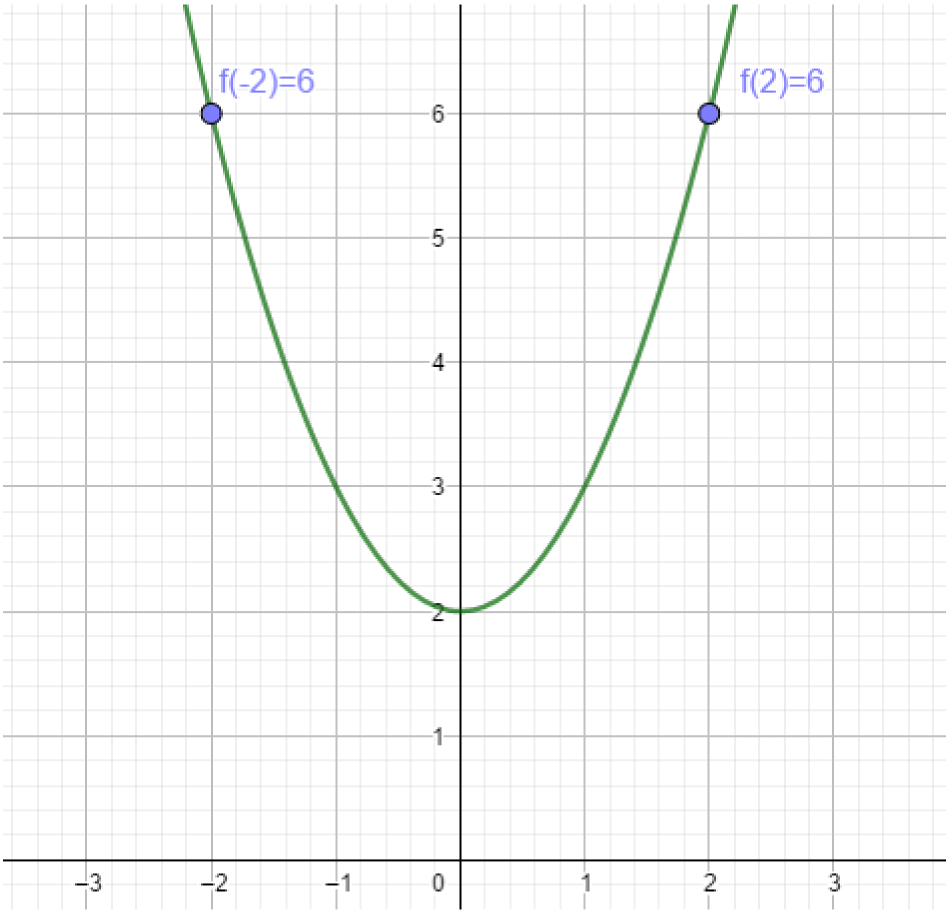
3.1 Symmetry

Regarding symmetry, we consider on the one hand the axis symmetry with respect to the y-axis and the point symmetry to the origin.

Axis symmetry

A function is axis symmetric to the y-axis if the graph on the left side of the y-axis is a mirror image of the graph on the right side of the y-axis, v.v. (vice versa or vice versa the same way).

It then follows f(x) = f(-x). You can see from the graph that f(2) = f(-2) = 6, for example.

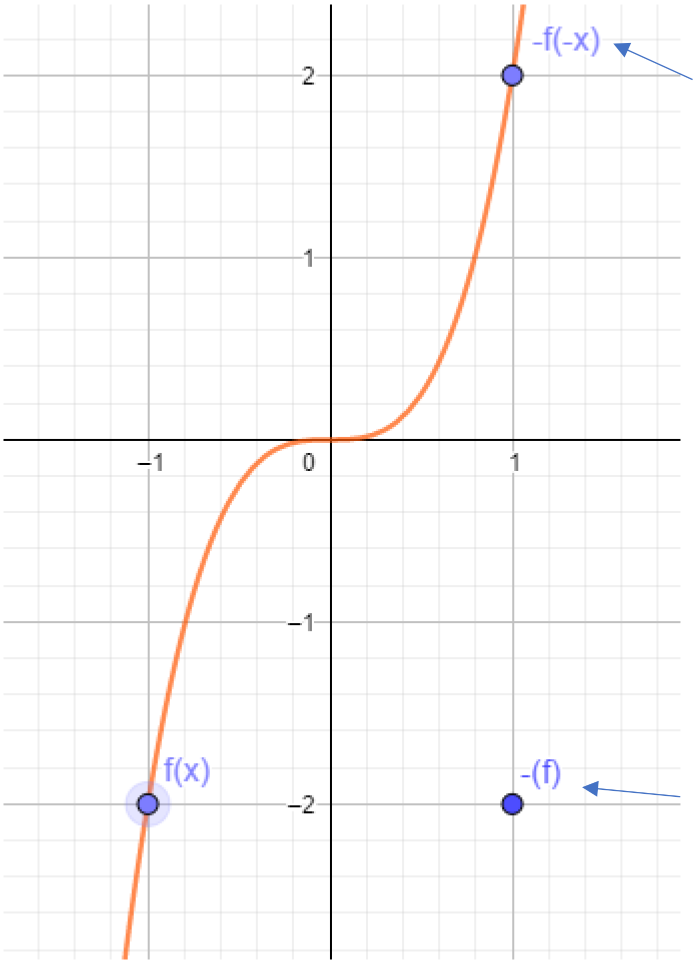


Coming with a completely rational function

f(x) = $a_n x^{n+1} + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$ only even exponents before, then the corresponding function is axisymmetric to the y-axis. The absolute term a0 may also be present. This moves the previously axisymmetric graph with respect to the y-axis.

<u>Point</u> symmetry

Point symmetry to the origin exists if f(-x) = -f(x).



-f(x) is the reflection of f(x) on the y axis.

-f(x) is the reflection of f(x) on the y-axis

f(-1) = -2 and f(1) = 2. You calculate –f(x) = –1 · f(x) = –1 · 2 = –2.
You can see from this example that f(-x) = -f(x).

For a completely rational function

f(x) = $a_n x^{n+1} + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$ only odd exponents before, then the corresponding function is point symmetric to the origin.

3.2. Profile tasks

If certain points are given by a function, the corresponding function rule can be determined from these points.

Write down the corresponding general function rule for the function of the given points.

A completely rational function of degree n has the following general function rule:

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$

Example:

If points A(4/5) and B(8/11) are given by a linear function, a first-degree function, the corresponding function rule is $f(x) = a_1x + a_0 = = mx + b$

From these two points we read the following information: $x_1 = 4$, $y_1 = 5$ and $x_2 = 8$, $y_2 = 11$.

The slope is calculated as follows: $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$
In order to calculate m one now inserts the two coordinate pairs into the above formula and obtains: $m = \frac{5-11}{4-8} = \frac{-6}{-4} = 1.5$ If one inserts m and one of the coordinate pairs in y = mx + b, one can calculate b from the resulting linear equation.

$5 = 1{,}5 \cdot 4 + b \mid -6$
$-1 = b$

The linear function rule belonging to points A(4/5) and B(8/11) is therefore
$f(x) = 1.5\,x - 1$.

To be on the safe side you can now carry out the point sample. For this one inserts one of the coordinate pairs in $f(x) = y = 1.5\,x - 1$ and checks whether a true statement results.

$11 = 1{,}5 \cdot 8 - 1 \rightarrow$ true

If points are given by a second, third or higher degree function, then the function rule can be determined by dividing the given points into

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ and sets up a linear system of equations from which the coefficients can be determined.

Example:

In order to determine the functional rule of a quadratic function of the form
$f(x) = a_2x^2 + a_1x + a_0$, three points must be given.

The points P1(-1|-4), P2(1|4) and P3(2.5|-0.5) are given.

In the first step, we place points P1, P2 and P3 one after another in the general form $f(x) = a_2x^2 + a_1x + a_0$ [Note: y=f(x)]. This way we get a linear system of 3 equations:

P(x|y)		$y = a_2 \; x^2 + a_1x + a_0$	
$P_1(-1\|-4)$:	I	$-4 = a_2\cdot(-1)^2 + a_1\cdot(-1) + a_0$	$\rightarrow a_2 - a_1 + a_0 = -4$
$P_2(1\|4)$:	II	$4 = a_2\cdot 1^2 + a_1\cdot 1 + a_0$	$\rightarrow a_2 + a_1 + a_0 = 4$
$P_3(2{,}5\|-0{,}5)$:	III	$-0{,}5 = a_2\cdot 2{,}5^2 + a_1\cdot 2{,}5 + a_0$	$\rightarrow 6{,}25a_2 + 2{,}5\,a_1 + a_0 = -0{,}5$

The linear system of equations is now solved by means of the addition method.

We write down the equation with the largest numbers first.

I: $6{,}25a_2 + 2{,}5\,a_1 + a_0 = -0{,}5$

II: $a_2 + a_1 + a_0 = 4 \quad | \cdot (-6{,}25)$

III: $a_2 - a_1 + a_0 = -4 \quad | \cdot (-6{,}25)$

I: $6{,}25a_2 + 2{,}5\,a_1 + a_0 = -0{,}5$

II: $-3{,}75\,a_1 - 5{,}25a_0 = -25{,}5 \quad | \cdot 8{,}75$

III: $8{,}75\,a_1 - 5{,}25\,a_0 = 24{,}5 \quad | \cdot 3{,}75$

I: $6{,}25a_2 + 2{,}5\,a_1 + a_0 = -0{,}5$

II: $-3{,}75\,a_1 - 5{,}25a_0 = -25{,}5$

III: $-65{,}625 = -131{,}25$

Now solve the lowest equation after a_0 and add the solutions found step by step to the upper equations.

$-65{,}625\,a_0 = -131{,}25 \mid : (-65{,}625)$

$a_0 = 2$

a_0 in II: $-3{,}75\,a_1 - 5{,}25 \cdot 2 = -25{,}5 \mid +10{,}5$

$-3{,}75\,a_1 = -15 \mid : (-4)$

$a_1 = 4$

a_0 and a_1 in I: $6{,}25a_2 + 2{,}5 \cdot 4 + 2 = -0{,}5 \mid -12$

$6{,}25a_2 = -12{,}5 \mid : 6{,}25$

$a_2 = -2$

Thus the function rule is $f(x) = -2x^2 + 4x + 2$

(source: https://www.mathebibel.de/quadratische-funktionen-funktionsgleichung-bestimmen)

Video: https://youtu.be/cxjXZSVUS0o

Calculation tasks: Steckbriefaufgaben zu quadratischen Funktionen

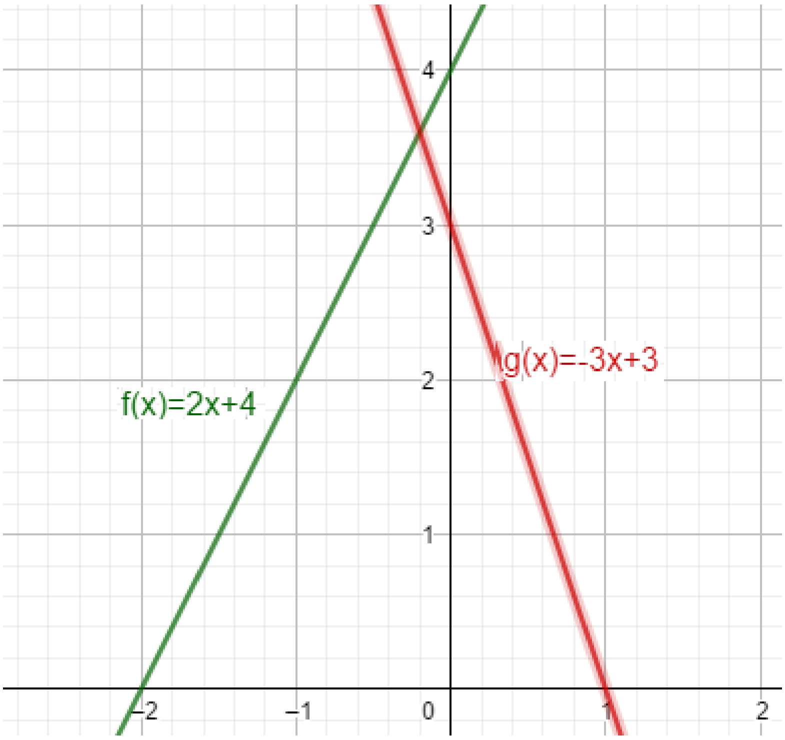
3.3 Intersections of functions

If you want to find the intersections of fully rational functions

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0$, both functions are set equal: $f(x) = g(x)$. Then the corresponding solution method is applied and the solutions found are inserted into f(x) or g(x) to calculate the corresponding y-coordinate.

Example:

$f(x) = 2x + 4$; $g(x) = -3x + 3$



$f(x) = g(x) \rightarrow 2x + 4 = -3x + 3 \mid -4$

$2x = -3x - 1 \quad \mid +3x$

$5x = -1 \quad \mid :5$

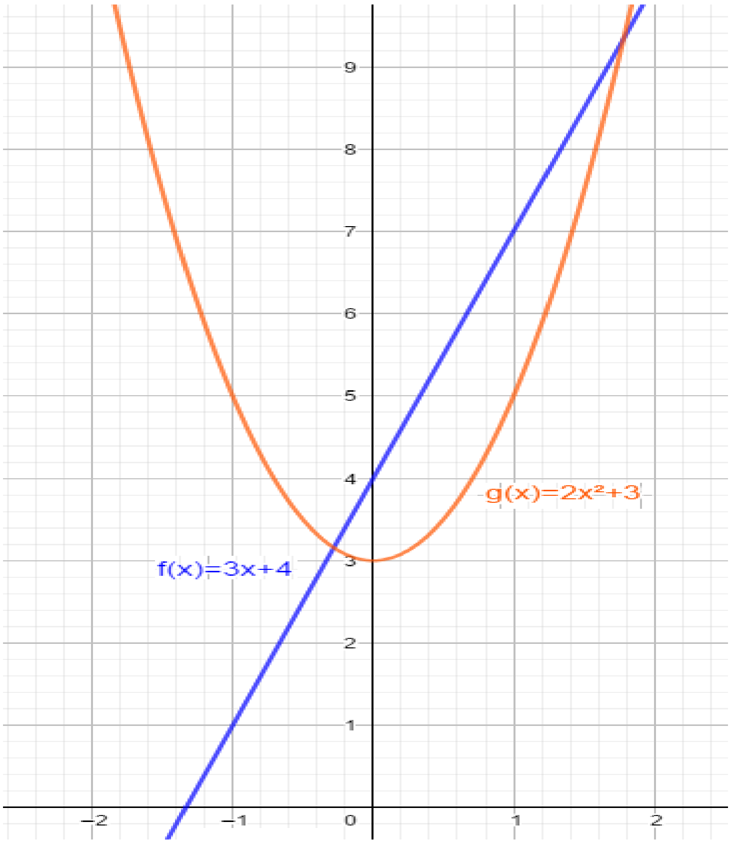
$x = -0{,}2$

Insert $x = -0,2$ in f(x) oder g(x) ein:
$f(-0,2) = 2 \cdot (-0,2) + 4 = 3,6$
The point of intersection thus has the coordinates (- 0.2 / 3.6)

You can also determine the intersection of a quadratic and a linear function.

Example:

f(x) $3x + 4$; g(x) $2x^2 + 3$



$f(x) = g(x) \rightarrow 3x + 4 = 2x^2 + 3$
Here you can see a quadratic equation, which can be solved with the p-q-formula.

However, two conditions must be met:

1. The equation must be rearranged so that one side of the equation says 0.

2. x^2 must be preceded by 1.

$3x + 4 = 2x^2 + 3 \quad | -2x^2 \quad | -3$

$-2x^2 + 3x + 1 = \quad | : (-2)$

$1x^2 - 1,5x - 0,5 = 0 \quad \rightarrow p = -1,5; q = -0,5$

Now apply the p-q formula to determine the two solutions.

$$x_{1/2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$x_{1/2} = -\frac{-1{,}5}{2} \pm \sqrt{\left(\frac{-1{,}5}{2}\right)^2 - (-0{,}5)} = 0{,}75 \pm \sqrt{\frac{17}{16}}$$

$x_1 = 1{,}7808$; $x_2 = -0{,}2808$

Insert now x_1 and x_2 in f(x) or g(x):

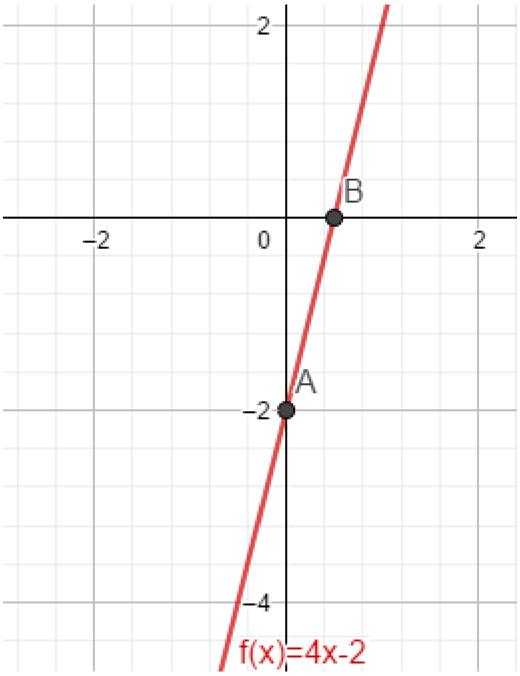
$f(1{,}7809) = 3 \cdot 1{,}7808 + 4 = 9{,}34 \rightarrow S_1(1{,}7809 / 9{,}34)$

$f(-0{,}2808) = 3 \cdot (-0{,}2808) + 4 = 3{,}16 \rightarrow S_2(-0{,}2808 / 3{,}16)$

3.4 Points of intersection with the axes

We are often interested in the points of intersection of the function graph with the axes, especially with the x-axis.

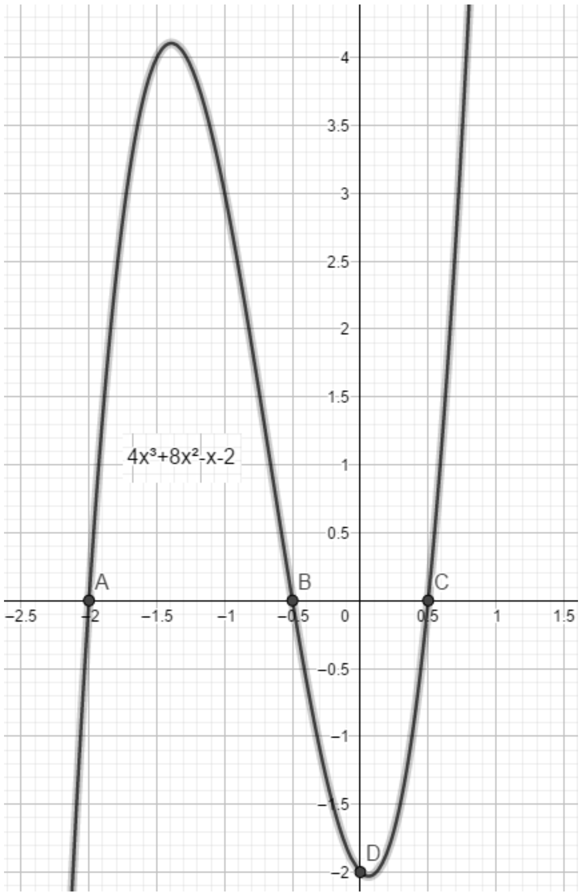
The graph of f(x) = 4x - 2 has an intersection with the x - axis and the y - axis



The graph of $f(x) = 4x^3 + 8x^2 - x - 2$

also has only one point of intersection with the y-axis, but three points of intersection with the x-axis.

Since one does not move in x - direction for intersection with the y - axis, this intersection point has the coordinate Sy(0 / ?) in any case. To determine the y - coordinate set x = 0, what follows:



f $f(0) = 4 \cdot 0^3 + 8 \cdot 0^2 - 0 – 2 = – 2 = y$.
➔ $S_y(0 / –2)$.
In general, the point of intersection with the

y - axis the coordinates $S_y(0 / f(0))$.

The points of intersection with the x-axis are the so-called zero points. Since they are located in y- direction is not moved, it follows that y = 0. To determine the points of intersection with the x-axis, set the function = 0.

$y = 0 = 4x^3 + 8x^2 - x - 2$.

The first zero point is determined using polynomial division. The other two by means of the

q - p - formula. The points of intersection with the x - axis are thus $S_{x1}(-2 / 0)$; $S_{x2}(-0.5 / 0)$; Sx3(0.5 / 0).

3.5 Parallel straight lines

Two straight lines can be in the plane or in space

- be identical
- be parallel to each other or
- have a point of intersection.

They are identical if all points of one line are also points of the other line.

The point of intersection of two lines is determined by equating the two functional equations, as can be read in chapter 3.2.

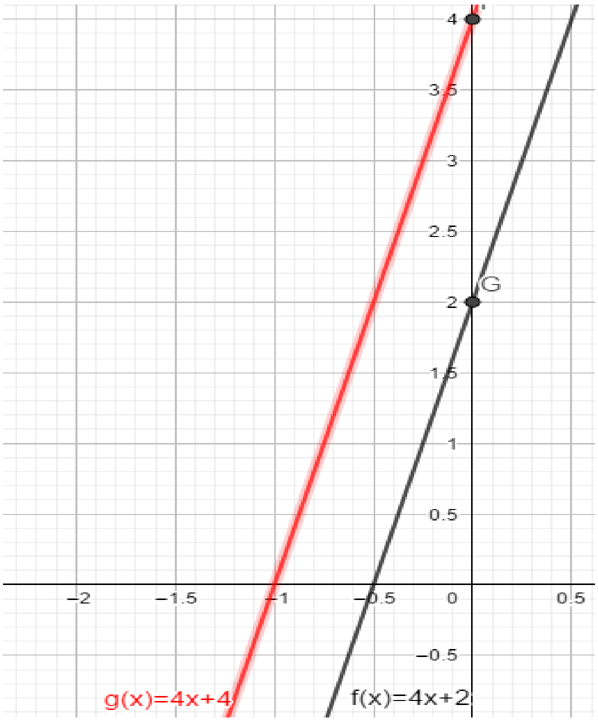
Parallel straight lines have the property that they have the same distance to each other everywhere. This is exactly the case when their gradients match.

As can be seen in the example

f(x) = 4x+4 and g(x) = 4x + 2 the slope is the same, namely $m_f = m_g = 2$.

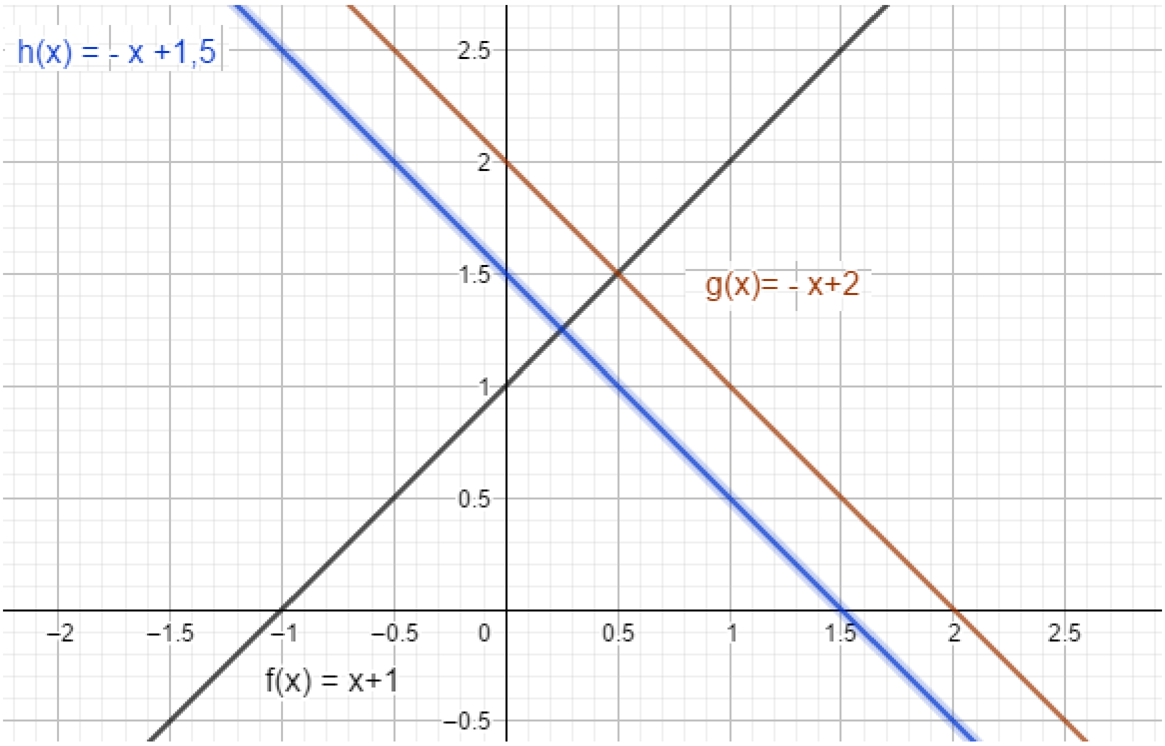
The y - axis intercept is different.

In general, two straight lines are parallel when $m_f = m_g$ und $b_f \neq b_g$



3.6 Orthogonal straight lines

To a straight line, you can calculate the orthogonal (vertical, right-angled) line.

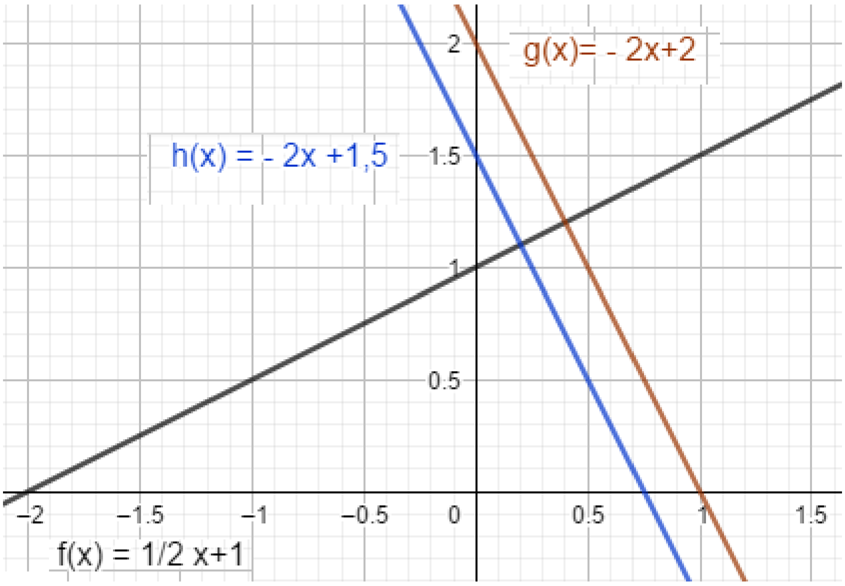


Example:

Given is the graph for f(x) = x + 1. Orthogonal straight lines have e.g. the function rule g(x) = - x + 2 or h(x) = - x + 1.5. You can see that the gradient of the orthogonal lines have a negative sign. The y axis intercept is arbitrarily selected.

But how do you determine exactly the gradient of an orthogonal straight line?

In the following example the function f(x) = $\frac{1}{2}x + 1$ is given.



The orthogonal straight lines have the function rule g(x) = -2x + 1.5 or h(x) = -2x + 2. You can see here too that the parameter b can be chosen at will. The gradient is the negative inverse of the original gradient.

The slope of f was $m_f = \frac{1}{2}$ The gradient of a line orthogonal to it is the negative sweep fraction thereof. So $m_g = -\frac{1}{\frac{1}{2}} = -1 : \frac{1}{2} = -2$

In general, $m_g = -\frac{1}{m_f}$ bzw. $m_g \cdot m_f = -1$

Thus we can check whether two straight lines are orthogonal to each other.

Example:

Check if the lines f(x) = - 0.6x + 2 and f(x) = $\frac{5}{3}x - 4$ are perpendicular to each other.

Insert the two slopes in $m_g \cdot m_f = -1$ and check:

$-0{,}6 \cdot \frac{5}{3} = -\frac{6}{10} \cdot \frac{5}{3} = -\frac{30}{30} = -1$. Thus it is shown that the two straight lines are perpendicular to each other.

If a straight line is given and a point through which a line orthogonal to it is to run, the equation can determine this line by calculation.

Example:

A straight line with the equation f(x) = 0.7 x + 1 is given. A line orthogonal to it is to run through the point P (2 / -3). First determine the gradient using
$m_g = -\frac{1}{m_f}$. $m_g = -\frac{1}{0,7} = -\frac{10}{7}$.
Insert now $m_g = -\frac{10}{7}$ in $g(x) = m_g x + b_g$.
$-3 = -\frac{10}{7} \cdot 2 + b_g \mid + \frac{20}{7}$
$b_g = -\frac{1}{7}$

Therefore $g(x) = -\frac{10}{7}x - \frac{1}{7}$